Maths Learning Service: Revision

Matrices

Intro. to Fin. Maths I

A matrix is an array of numbers, written within a set of [ ] brackets, and arranged into a pattern of rows and columns. For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 21 & 7 & -4 & 9 \end{bmatrix}$$

The **order** (or **size**, or **dimension**) of a matrix is written as "$m \times n$" where $m =$ the number of rows, and $n =$ the number of columns. For example, the matrices above have dimensions

$$2 \times 3, \quad 3 \times 3 \quad \text{and} \quad 1 \times 4.$$

Basic Matrix Operations

Addition (or subtraction) of matrices is performed by adding (or subtracting) elements in corresponding positions. Addition is only valid if the two matrices have the same order.

Examples:

(i) $$\begin{bmatrix} 2 & -4 & 0 \\ -1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2+3 & -4+4 & 0+(-1) \\ -1+7 & 3+0 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 5 & 0 & -1 \\ 6 & 3 & 3 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 7 \\ -8 & 1 \end{bmatrix} = \begin{bmatrix} 3-1 & 4-7 \\ -2-(-8) & 0-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 6 & -1 \end{bmatrix}$$

(iii) $\begin{bmatrix} 2 & -4 & 0 \\ -1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -2 & 0 \end{bmatrix}$ cannot be done as the orders are different.

When a matrix is multiplied by a real number (called a *scalar*), each element is multiplied by the scalar. The result is another matrix of the same order.

Examples:

(i) $$4\begin{bmatrix} 2 & 1 \\ -3 & 9 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 4 \times 2 & 4 \times 1 \\ 4 \times -3 & 4 \times 9 \\ 4 \times 0 & 4 \times -5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -12 & 36 \\ 0 & -20 \end{bmatrix}$$

(ii) $$\frac{1}{2}\begin{bmatrix} 7 & 8 & -10 & 6 & 0.4 \end{bmatrix} = \begin{bmatrix} 3.5 & 4 & -5 & 3 & 0.2 \end{bmatrix}$$

(iii) $$2\begin{bmatrix} 5 & -3 \\ 0 & -6 \end{bmatrix} - 3\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ 0 & -12 \end{bmatrix} - \begin{bmatrix} 9 & 12 \\ -3 & 21 \end{bmatrix} = \begin{bmatrix} 1 & -18 \\ 3 & -33 \end{bmatrix}$$

When giving matrices a name, use capital letters such as A, B, etc to distinguish them from algebraic scalars such as a, b, etc.

Exercises

(1) Given that

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 1 & -3 \\ 2 & 0 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ -4 & 9 \end{bmatrix} \quad D = \begin{bmatrix} 11 & 5 \\ 0 & -2 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}$$

find the following (if possible):

(a) $A + B$ (b) $B + A$ (c) $C + D$ (d) $C - D$ (e) $D - C$

(f) $A + E$ (g) $B - D$ (h) $3A$ (i) $2C + D$ (j) $5B - 4E$

Matrix Multiplication

The rule for multiplying matrices can be represented as follows:

$$AB = \begin{bmatrix} \text{row 1 of } A \times \text{col 1 of } B & \text{row 1 of } A \times \text{col 2 of } B & \text{row 1 of } A \times \text{col 3 of } B & \cdots \\ \text{row 2 of } A \times \text{col 1 of } B & \text{row 2 of } A \times \text{col 2 of } B & \text{row 2 of } A \times \text{col 3 of } B & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where "row i of $A \times$ col j of B" is a single number and stands for "each entry in row i of A is multiplied by the corresponding entry in column j of B and the results are added together".

Examples:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -5 \\ -1 & -2 \\ 0 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(i) $$CA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times (-1) + 2 \times 4 & 1 \times 3 + 2 \times 5 \\ 3 \times 2 + 4 \times 1 & 3 \times (-1) + 4 \times 4 & 3 \times 3 + 4 \times 5 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 13 \\ 10 & 13 & 29 \end{bmatrix}$$

(ii) $$AB = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + (-1) \times (-1) + 3 \times 0 & 2 \times (-5) + (-1) \times (-2) + 3 \times 3 \\ 1 \times 4 + 4 \times (-1) + 5 \times 0 & 1 \times (-5) + 4 \times (-2) + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 0 & 2 \end{bmatrix}$$

(iii) $$CB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -1 & -2 \\ 0 & 3 \end{bmatrix}$$

This is not possible because there are fewer entries in the rows of C (two) than in the columns of B (three).

Matrix multiplication is only defined when the number of columns in the first matrix equals the number of rows in the second.

(iv) $CD = \begin{bmatrix} 13 & 19 \\ 27 & 43 \end{bmatrix}$ but $DC = \begin{bmatrix} 16 & 22 \\ 27 & 40 \end{bmatrix}$ so $CD \neq DC$.

In general $AB \neq BA$ for matrices.

(v)
$$CI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times0 & 1\times0+2\times1 \\ 3\times1+4\times0 & 3\times0+4\times1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = C \text{ (unchanged)}$$

(vi)
$$IC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = C \text{ (unchanged)}$$

The matrix I is an identity matrix and is the matrix equivalent of the number 1 in scalar multiplication.

Notes:

1. The identity is an exception to the general rule for matrix multiplication since $CI = IC = C$.
2. Identity matrices only exist for square matrices. The matrix I used in Examples (v) and (vi) is called "the identity matrix for a 2×2 matrix". The identity matrix for a 3×3 matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercises

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 4 \end{bmatrix} \quad E = \begin{bmatrix} -3 & 2 \\ 1 & 7 \end{bmatrix}$$

(2) Using the above matrices, calculate the following (if possible):

(a) AB (b) BA (c) DI (d) ID (e) CD

(f) DC (g) BC (h) CB (i) E^2 (j) B^2

Answers to Exercises

(1) (a) $\begin{bmatrix} 6 & 3 & -3 \\ 6 & 5 & 9 \end{bmatrix}$ (b) same as (a) (c) $\begin{bmatrix} 12 & 7 \\ -4 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} -10 & -3 \\ -4 & 11 \end{bmatrix}$

(e) $\begin{bmatrix} 10 & 3 \\ 4 & -11 \end{bmatrix}$ (f) not possible (g) not possible (h) $\begin{bmatrix} -3 & 6 & 0 \\ 12 & 15 & 9 \end{bmatrix}$

(i) $\begin{bmatrix} 13 & 9 \\ -8 & 16 \end{bmatrix}$ (j) not possible

(2) (a) $\begin{bmatrix} 2 & 3 & 1 \\ -8 & -5 & -5 \end{bmatrix}$ (b) not possible (c) D (d) D

(e) not possible (f) $\begin{bmatrix} 3 \\ 6 \\ 15 \end{bmatrix}$ (g) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ (h) not possible

(i) $E^2 = EE = \begin{bmatrix} 11 & 8 \\ 4 & 51 \end{bmatrix}$ (j) not possible